Problems with generalising: Pythagoras in N dimensions

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A post-16 mathematics student, Peter (not his real name), asked me, "Does Pythagoras' theorem generalise to three dimensions?"

I said, "Yes." But it turned out that we had different things in mind.

Peter was familiar with finding the modulus of two-dimensional vectors such as (3, 4) by working out $\sqrt{3^2+4^2}=5$, so when faced with the three-dimensional vector (3, 4, 5) it seemed to him like a natural extension to calculate the magnitude as $\sqrt[3]{3^3+4^3+5^3}=6$. He had generalised from $a^2 + b^2 = c^2$ in two dimensions to $a^3 + b^3 + c^3 = d^3$ in three dimensions, whereas what I had in mind was $a^2 + b^2 + c^2 = d^2$.

Pythagoras' theorem in two and three dimensions appears in General Mathematics, Units 1–2, section 6 (Geometry and trigonometry: Shape and measurement) in the *Victorian Certificate of Education Mathematics Study Design* (Victorian Curriculum Assessment Authority, 2010). It also comes in Further Mathematics, Units 3–4 (Applications: Geometry and trigonometry) in the same document. In the UK (and in Australia), students typically meet the two-dimensional version when aged around 13–14 and become quite familiar with this before encountering the three-dimensional version later on. Pythagoras' theorem is likely to be a significant element in any secondary mathematics curriculum, and is one of the topics adults frequently recall when talking about their experiences of school mathematics.

In the discussion that followed our initial conversation, several factors emerged as important. Peter had a strong sense that two dimensions are to do with *area* and *squaring* whereas three dimensions are related to *volume* and *cubing*. The move into three dimensions therefore led him logically to cubes rather than squares. Dimensionally, his rule was fine, since the cube root of the cube of a length is a length. In addition, the fact that the answer coincidentally came to an integer gave him extra confidence that all was well—he even had a nice arithmetic sequence: 3, 4, 5 in the first case; 3, 4, 5, 6 in the second. Peter seemed to have acted in highly mathematical ways, thinking 'like a mathematician', building on what he knew to form a conjecture about something else; it was just unfortunate that his conclusion happened to be

wrong. I was struck by the logic of what Peter felt would happen when ‘going up a dimension’ and the plausibility of his conjecture. Had it been correct, it would have been the kind of result that a teacher might have presented to a class with little justification—a mere wave of the hand and the words ‘by analogy’. So I am always rather disturbed when a potentially ‘obvious’ result is not right.

I often try to encourage students to generalise in mathematics lessons, as I see it as an extremely important action for them to take. As Mason (1996) expresses it:

> Generalization is the heartbeat of mathematics, and appears in many forms. If teachers are unaware of its presence, and are not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place. (p. 65)

Yet here the generalisation had gone off in an erroneous direction. The neatness of the integer answer that Peter had obtained also made me question whether students are too often offered ‘nice’ examples that are intended to give them reassurance, and that perhaps this is not always a helpful practice. In this case, however, the integer answer was a complete fluke, but possibly this situation might have been anticipated and deliberately avoided by the question writer. On the other hand, I am very glad that the problem did arise, as it led to such an interesting encounter.

One way to imagine Pythagoras’ theorem in a third dimension is to extend the right-angled triangle (with hypotenuse c and legs a and b) into a right prism of length l, the squares on the sides of the triangle becoming cuboids on the faces of the prism as shown in Figure 1, where the right-angled triangle is shown in black. Then, Pythagoras’ theorem is exactly equivalent to saying that the *volumes* of the *cuboids* on the two smaller rectangular faces sum to the volume of the cuboid on the largest face:

$$a^2 l + b^2 l = c^2 l \Leftrightarrow a^2 + b^2 = c^2,\ l > 0$$

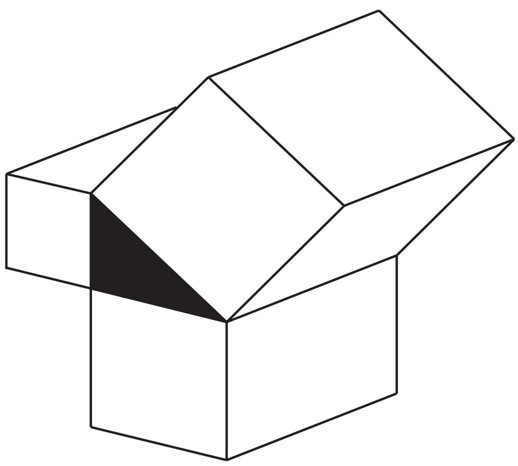

Figure 1. Pythagoras’ theorem in a third dimension.

I did not simply want to tell Peter that his idea was wrong. Instead, I sought to explore with him the consequences of his conjecture, so that he might perhaps meet some kind of cognitive conflict; something that would unsettle him and cause him to doubt and thus probe what he had done. Since we had been considering two and three dimensions, I first tried to envisage what Peter's pattern would lead to in '*one* dimension' as shown in Table 1. It would seem to be $a = b$, but that did not help to establish a conflict any more than venturing into four dimensions might.

Table 1. Extending Pythagoras' theorem to one dimension.

number of dimensions	Peter's equation
3	$a^3 + b^3 + c^3 = d^3$
2	$a^2 + b^2 = c^2$
1	?

Instead I asked Peter to find the magnitude of the vector (3, 4, 0). He realised before doing any calculations that this should give the same answer as the magnitude of (3, 4), since (3, 4, 0) is just another way of writing (3, 4). At this point, the fact that $\sqrt[3]{3^3 + 4^3 + 0^3} \neq 5$ persuaded him that something was wrong with his generalisation. I wanted him to see a problem with what he had done before offering him an alternative. I then drew out a 3–4–5 triangle and extended the diagram into a cuboid (Figure 2), helping Peter to see that the space diagonal was $5\sqrt{2} > 6$. This revealed that cubing, perhaps surprisingly, does not actually come into it at all.

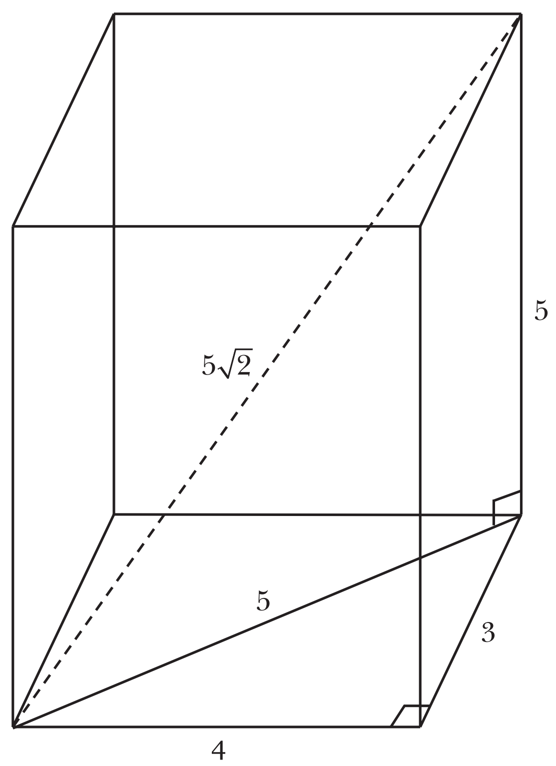

Figure 2. Challenging Peter's conjecture.

Afterwards, I thought more about this episode. In addition to *Pythagorean triples*, such as (3, 4, 5), which satisfy the equation $a^2 + b^2 = c^2$ in positive integers, I was also aware of a few *Pythagorean quadruples*, such as (2, 3, 6, 7), which satisfy the equation $a^2 + b^2 + c^2 = d^2$ in positive integers. Since Peter had stumbled across $3^3 + 4^3 + 5^3 = 6^3$, I looked for other solutions to the Diophantine equation $a^3 + b^3 + c^3 = d^3$, and found that it was Euler who discovered the complete solution, with Ramanujan later finding a simpler form (see Berndt & Bhargava, 1993).

I then began to wonder whether it was possible to find a set of three positive integers a, b, c such that:

$$a^2 + b^2 + c^2 = d^2$$

and

$$a^3 + b^3 + c^3 = e^3$$

where d and e are integers. Such a vector (a, b, c) would give an integral value for the 'magnitude' whether calculated correctly or by Peter's cubing-and-cube-rooting method. In fact, this is possible, four examples which do this being (3, 34, 114), (14, 23, 70), (18, 349, 426) and (145, 198, 714). These examples, known as *Martin triples*, are given at http://sites.google.com/site/tpiezas/011 (see number 12; accessed 30 September 2012), which reports the early twentieth-century work of Artemas Martin.[1] The mathematics is demanding, but if negative integers are also allowed then there is a parametrisation.

It is easy for teachers to talk enthusiastically about the importance of students generalising their mathematics, but this will not inevitably lead to correct results unless carried out with sufficient thought. In many cases, students can generalise a given result in any number of different directions, not all of which may be mathematically accurate. What happens when 'going up a level' may not be straightforward, and generalisations, like anything else, must be treated as conjectures until proved.

References

Allaire, P. R. & Cupillari, A. (2000). Artemas Martin: An amateur mathematician of the nineteenth century and his contribution to mathematics. *The College Mathematics Journal, 31*(1), 22–34

Berndt, B. C. & Bhargava, S. (1993). Ramanujan for lowbrows. *American Mathematical Monthly, 100*, 644–656

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds), *Approaches to algebra* (pp. 65–86). Dordrecht/Boston/London: Kluwer.

Victorian Curriculum Assessment Authority [VCAA]. (2010). *Victorian Certificate of Education Mathematics Study Design*. Retrieved 30 September 2012 from http://www.vcaa.vic.edu.au/Documents/vce/mathematics/mathsstd.pdf

1 For more information about Artemas Martin, see www.american.edu/library/archives/finding_aids/martin_fa.cfm (accessed 30 September 2012) and also Allaire and Cupillari (2000).